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ψ-UNIFORM STABILITY FOR LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we show the ψ-uniform stability for linear impulsive differential equations and their perturbations by using the impulsive Gronwall's inequalities.

1. Introduction

Impulsive differential equations and application were introduced by some authors: A. M. Samoilenko and N. A. Perestyuk [3], V. Lakashmikantham, D. D. Bainov and P. S. Simeonov[13], Bainov and Simeonov[4, 6].

Recently, it has been realized that impulsive differential equations form a natural description of observed evolution phenomena of several real world problems, and therefore their study has attracted much attention [13].

Akinyele[1] introduced the notion of $\psi-$stability of degree k with respect to an increasing function $\psi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, which is differentiable on $\mathbb{R}_+$ and such that $\psi(t) \geq 1$ for $t \geq t_0$ and $\lim_{t\to\infty} \psi(t) = b,\ b \in [1, \infty)$. Diamandescu in[2], proved some sufficient conditions for $\psi-$stability of the zero solution of a nonlinear Volterra integro-differential system. Bhanu Gupta and Sanjay K. Srivastava investigated $\psi-$ exponential stability of non-linear impulsive differential equaions[12].

In this paper we show the ψ-uniform stability for linear impulsive differential equations at fixed moments and their perturbations by using impulsive inequality of Gronwall type.

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2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and $\|\cdot\|$ denotes the norm on $\mathbb{R}^n$.

Let $\nu = \{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ be an unbounded and increasing sequence. Denoted by $PC([t_0, \infty), \mathbb{R}^n \times \mathbb{R}^n)$ the set of functions $\varphi : [t_0, \infty) \to \mathbb{R}^n \times \mathbb{R}^n$ which are continuous for $t \in [t_0, \infty) \setminus \nu$, are continuous from the left for $t \in [t_0, \infty)$, and have discontinuities of the first type at the points t_k for each $k \in \mathbb{N}$.

We consider the linear impulsive system

$$\begin{cases} x' = A(t)x, & t \neq t_k, \\ \Delta x = B_k x, & t = t_k, \\ x(t_0^+) = x_0, \end{cases} \tag{2.1}$$

where $A \in PC([t_0, \infty), \mathbb{R}^n \times \mathbb{R}^n)$, and its perturbed linear system with fixed moments of impulse

$$\begin{cases} y' = A(t)y + C(t)y, & t \neq t_k, \\ \Delta y = B_k y + R_k y, & t = t_k, \\ y(t_0^+) = y_0, \end{cases} \tag{2.2}$$

where $C \in PC([t_0, \infty), \mathbb{R}^n \times \mathbb{R}^n)$, and B_k, R_k are $n \times n$ matrices.

We assume that the solution $y(t)$ of system (2.2) is left continuous at the moments of impulsive effect t_k, i.e., $y(t_k^-) = y(t_k)$, and $\Delta y(t_k) = y(t_k^+) - y(t_k)$.

Lemma 2.1. [5, Theorem 1.5] *Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Then the following statements hold:*

1. *There exists a unique solution of equation (2.1) with $x(t_0^+) = x_0$ (or $x(t_0) = x_0$) and this solution is defined for $t > t_0$ (or $t \geq t_0$).*
2. *If $det(E + B_k) \neq 0$ for each $k \in \mathbb{Z}$, then this solution is defined for all $t \in \mathbb{R}$.*

The next result follows from a simple calculation.

Lemma 2.2. [5] *Each solution $y(t)$ of (2.2) satisfies the integro-summary equation*

$$y(t) = W(t, s^+)y(s) + \int_s^t W(t, \tau)C(\tau)y(\tau)d\tau + \sum_{s<t_k<t} W(t, t_k^+)R_k y(t_k), \ t \geq s,$$

where $W(t, s)$ is the Cauchy matrix for equation (2.1).

LEMMA 2.3. [5, Lemma 1.4] *suppose that for* $t \geq t_0$ *the inequality*

$$u(t) \leq c + \int_{t_0}^{t} b(s)u(s)ds + \sum_{t_0 \leq t_k < t} \beta_k u(t_k) \tag{2.3}$$

holds, where $u \in PC(\mathbb{R}, \mathbb{R})$, $b \in PC(\mathbb{R}, \mathbb{R}^+)$ *and* $\beta_k \geq 0$, $k \in \mathbb{Z}$ *and* c *are constants. Then we have*

$$u(t) \leq c \prod_{t_0 \leq t_k < t} (1 + \beta_k) \exp\Big(\int_{t_0}^{t} b(s)ds\Big) \tag{2.4}$$

$$\leq c \exp\Big(\int_{t_0}^{t} b(s)ds + \sum_{t_0 \leq t_k < t} \beta_k\Big), \quad t \geq t_0. \tag{2.5}$$

We will prove that under a general "small" mean condition on the perturbations C and R_k, ψ-uniform stability of system (2.1) is inherited by the perturbed system (2.2).

DEFINITION 2.4. [8, Definition 2.5] The zero solution $\upsilon = 0$ of (2.1) (or system (2.1)) is called ψ-uniform stable if there exist a finite $\gamma > 0$ and invertible matrix function $\psi(t) \in PC([t_0, \infty), \mathbb{R}^n \times \mathbb{R}^n)$(or left continuous function $\psi(t) \in PC([t_0, \infty), \mathbb{R}^+)$) such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|\psi(t)x(t)\| \leq \gamma \|\psi(t_0)x(t_0)\|, \ t \geq t_0 > 0. \tag{2.6}$$

If we choose a convenient value of t , then we see that $\psi(t)$ is reduced to the unit matrix of order n. It is easy to see that if $\psi(t)$ is unit matrix, then the ψ-uniform stability is equivalent with the uniform stability.

3. Main results

THEOREM 3.1. *The linear impulsive system (2.1) is* ψ*-uniform stable if and only if there exists a* $\gamma > 0$ *and invertible matrix function* $\psi(t) \in PC([t_0, \infty), \mathbb{R}^n \times \mathbb{R}^n)$ *such that for every* $x_0 \in \mathbb{R}^n$,

$$\|\psi(t)W(t, t_0)\psi^{-1}(t_0)\| \leq \gamma, \ t \geq t_0 > 0, \tag{3.1}$$

where $W(t, t_0)$ *is the Cauchy matrix of (2.1).*

Proof. Suppose that (2.1) is $\psi-$uniformly stable. Then, there is a $\gamma > 0$ such that for any $t_0, x(t_0)$, the solutions satisfy

$$\|\psi(t)x(t)\| \leq \gamma \|\psi(t_0)x(t_0)\|, \ t \geq t_0.$$

Given any t_0 and $t_a \geq t_0$, let x_a be a vector such that $\|\psi(t_0)x_a\| = 1$

$$\begin{aligned}\|\psi(t_a)W(t_a,t_0)x_a\| &= \|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\psi(t_0)x_a\| \\ &= \|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\|\|\psi(t_0)x_a\| \\ &= \|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\|.\end{aligned}$$

So the initial state $x(t_0) = x_a$ gives a solution of (2.1) so that time t_a satisfies

$$\begin{aligned}\|\psi(t_a)x(t_a)\| &= \|\psi(t_a)W(t_a,t_0)x_a\| \\ &= \|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\|\|\psi(t_0)x_a\| \\ &= \|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\| \\ &\leq \gamma\|\psi(t_0)x(t_0)\|.\end{aligned}$$

Since $\|\psi(t_0)x_a\| = 1$, we see that $\|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\| \leq \gamma$. Since x_a can be selected for any t_0 and $t_a \geq t_0$, we see that $\|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\| \leq \gamma$ for all $t, t_0 \in \mathbb{R}$.

Now suppose that there exists a γ such that $\|\psi(t_a)W(t_a,t_0)\psi^{-1}(t_0)\| \leq \gamma$ for all $t, t_0 \in \mathbb{R}$. For any t_0 and $x(t_0) = x_0$, the solution of (2.1) satisfies

$$\begin{aligned}\|\psi(t)x(t)\| &= \|\psi(t)W(t,t_0)x(t_0)\| \\ &= \|\psi(t)W(t,t_0)\psi^{-1}(t_0)\|\|\psi(t_0)x_0\| \\ &\leq \gamma\|\psi(t_0)x(t_0)\|, t \geq t_0.\end{aligned}$$

Thus, $\psi-$uniform stability of (2.1) established. □

THEOREM 3.2. *If the zero solution $x = 0$ of (2.1) is ψ-uniformly stable and there exists a constant M such that*

$$\int_0^\infty \|\psi(\tau)C(\tau)\psi(\tau)^{-1}\|d\tau + \sum_{0\leq t_k\leq\infty} \|\psi(t_k^+)R_k\psi^{-1}(t_k)\| \leq M \tag{3.2}$$

then the zero solution $y = 0$ of (2.2) is ψ-uniformly stable.

Proof. It follows from Lemma 2.2 that the solution $y(t)$ of (2.2) is given by

$$y(t) = W(t,t_0^+)y_0 + \int_{t_0}^t W(t,s)C(s)y(s)ds + \sum_{t_0<t_k<t} W(t,t_k)R_ky(t_k),\ t \geq t_0.$$

Then by Theorem3.1 there exist a constant $\gamma > 0$ such that

$$\|\psi(t)W(t,t_0)\psi^{-1}(t_0)\| \leq \gamma,\ t \geq t_0 \geq 0,$$

where $W(t,t_0)$ is the cauchy matrix of (2.1). Thus we obtain

$$
\begin{aligned}
\psi(t)y(t) = {} & \psi(t)W(t,t_0)\psi^{-1}(t_0)\psi(t_0)y(t_0) \\
& + \int_{t_0}^{t} \|\psi(t)W(t,\tau)\psi^{-1}(\tau)\psi(\tau)C(\tau)y(\tau)d\tau \\
& + \sum_{t_0<t_k<t} \psi(t)W(t,t_k^+)\psi^{-1}(t_k^+)\psi(t_k^+)R_k\psi^{-1}(t_k)\psi(t_k)y(t_k),
\end{aligned}
$$

$$
\begin{aligned}
\|\psi(t)y(t)\| \le {} & \gamma\|\psi(t_0)y(t_0)\| + \gamma\int_{t_0}^{t} \|\psi(\tau)C(\tau)\psi^{-1}(\tau)\|\|\psi(\tau)y(\tau)\|d\tau \\
& + \gamma\sum_{t_0<t_k<t} \|\psi(t_k^+)R_k\psi^{-1}(t_k)\|\|\psi(t_k)y(t_k)\|.
\end{aligned}
$$

By the Gronwall impulsive integral inequality [5]

$$
\begin{aligned}
\|\psi(t)y(t)\| & \le \gamma\|\psi(t_0)y(t_0)\|\exp[\gamma\int_{t_0}^{t} \|\psi(\tau)C(\tau)\psi^{-1}(\tau)\|d\tau \\
& \qquad + \gamma\sum_{t_0<t_k<t} \|\psi(t_k^+)R_k\psi^{-1}(t_k)\|],\ t \ge t_0. \\
& \le \gamma\|\psi(t_0)y(t_0)\|e^{\gamma M}, \\
& \le \gamma'\|\psi(t_0)y(t_0)\|
\end{aligned}
$$

where $\gamma' = \gamma e^{\gamma M}$. Hence the zero solution $y = 0$ of (2.2) is ψ-uniformly stable. The proof is complete. □

Corollary 3.3. *If we set $\psi(t) = 1/h(t)$, then the Theorem 3.2 is similar to Theorem 2.7 in* [10].

Corollary 3.4. *If the zero solution $x = 0$ of (2.1) is uniformly Lipschitz stable and there exists a constant M such that*

$$
\int_0^{\infty} C(\tau)d\tau + \sum_{0\le t_k\le\infty} R_k \le M \tag{3.3}
$$

then the zero solution $y = 0$ of (2.2) is uniformly Lipschitz stable.

In order to obtain ψ-uniformly stability of solutions of nonlinear impulsive differential systems, we need the following assumption.
(H):

(i) $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for every $y \in \mathbb{R}^n$, $n \in \mathbb{N}$

$$
\lim_{(t,x)\to(t_k,y)} f(t,x) \text{ exists for } t > t_k.
$$

In addition, there exists $\lambda \in PC(\mathbb{R}^+, \mathbb{R})$ such that

$$|f(t,y)| \leq \lambda(t)|y|, \; for \; (t,y) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

(ii) For every $k \in \mathbb{N}$, B_k is an $n \times n$ matrix, and $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and satisfies

$$|I_k(y)| \leq \lambda_k |y|, \; y \in \mathbb{R}^n, \lambda_k > 0,$$

We consider nonlinear impulsive differential system

$$\begin{cases} y' = A(t)y + f(t,y), \; t \neq t_k, \\ \Delta y = B_k y + I_k(y), \quad t = t_k, \\ y(t_0^+) = y_0, \end{cases} \tag{3.4}$$

where $f(t,0) = 0$.

We can obtain the various stability results from Theorem 3.2.

THEOREM 3.5. *Assume that conditions (H_1) holds, If the zero solution $x = 0$ of 2.1 is ψ-uniformly stable with condition H_2: $\int_0^\infty \|\psi(\tau)\lambda(\tau)\psi^{-1}(\tau)\| d\tau + \sum_{0 \leq t_k \leq \infty} \|\psi(t_k^+)\lambda_k \psi^{-1}(t_k)\| < \infty$, for each $t > 0$, then the zero solution $y = 0$ of (3.4) is ψ-uniformly stable.*

The proof of Theorem 3.5 can be proved in a similar manner as that of Theorem 3.2. So we omit the proof.

COROLLARY 3.6. *Assume that the ordinary differential system:*

$$x'(t) = A(t)x(t)$$

is ψ-uniformly stable. Furthermore, suppose that f, $I_k(k \in \mathbb{N})$ and $\psi(t)$ satisfy the hypothesis of Theorem 3.5. Then the impulsive system

$$\begin{aligned} y'(t) = A(t)y(t) + f(t,y), \; t \neq t_k, \\ \Delta y(t_k) = I_k(y), \; t = t_k. \end{aligned}$$

is ψ-uniformly stable.

To illustrate our results, we will give an example about ψ-uniformly stability of linear impulsive differential system.

EXAMPLE 3.7. *Consider the linear impulsive differential equation*

$$\begin{cases} x'(t) = \frac{1}{t^2}x, \; t \neq t_k, t > 0, \\ \Delta x = \frac{1}{k^2}x, \; t = t_k, k \in \mathbb{N} \end{cases} \tag{3.5}$$

where $\frac{1}{t^2} \in PC(\mathbb{R}^+, \mathbb{R})$, $\frac{1}{k^2} \in \mathbb{R}$, and $det(1 + \frac{1}{k^2}) \neq 0$ for each $k \in \mathbb{Z}^+$. Then the zero solution $x = 0$ of (3.5) is ψ-uniformly stable.

Proof. Let $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$. Then we have

$$W(t,t_0) = \prod_{t_0 \le t_k < t} (1 + \frac{1}{k^2}) \exp\Big(\int_{t_0}^{t} \frac{1}{\tau^2} d\tau\Big), \ t \ge t_0 > 0,$$

and furthermore

$$\begin{aligned} |W(t,t_0)| &= \Big| \prod_{t_0 \le t_k < t} (1 + \frac{1}{k^2}) \exp\Big(\int_{t_0}^{t} \frac{1}{\tau^2} d\tau\Big)\Big| \\ &\le \exp\Big(\int_{t_0}^{t} \frac{1}{\tau^2} d\tau + \sum_{t_0 \le t_k \le \infty} \Big|\frac{1}{k^2}\Big|\Big). \end{aligned}$$

it is easy to see that there exists $M > 0$ such that

$$\exp\Big(\int_{t_0}^{t} \frac{1}{\tau^2} d\tau + \sum_{t_0 \le t_k \le \infty} \Big|\frac{1}{k^2}\Big|\Big) \le M.$$

$$\begin{aligned} |\psi(t)W(t,t_0)\psi^{-1}(t_0)| &\le \Big|\psi(t) \exp\Big(\int_{t_0}^{t} \frac{1}{\tau^2} d\tau + \sum_{t_0 \le t_k \le \infty} \Big|\frac{1}{k^2}\Big|\Big)\psi^{-1}(t_0))\Big| \\ &\le \gamma. \end{aligned}$$

Where $\psi(t) = e^{-t}$ for $t \ge t_0 > 0$ and $\gamma = M$.
Hence the zero solution $x = 0$ of (3.5) is ψ-uniformly stable by Theorem 3.1. □

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